A LOG-NORMAL DIFFUSION PROCESS APPLIED TO THE DEVELOPMENT OF INDIAN AGRICULTURE WITH SOME CONSIDERATIONS ON ECONOMIC POLICY*

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Consider a stochastic variable say real agricultural production in India PER CAPITA. Then the transition probability that this variable will be y at time t if it was x at time s is given by :

$$(1)\quad f(s, x; t, y)=\frac{exp. \{-[log\ y-log\ x-b\ (t-s)]/2c\ (t-s)\}}{y\sqrt{2\pi c\ (t-s)}}.$$

This is a lognormal diffusion process which satisfies the forward and backward Kolmogoroff equations. In other words, it is the logarithm of REAL agricultural production PER CAPITA in India which is normally distributed and follows a conventional diffusion process.

Assume now that with probability one the variable x has the value x_0, at the point $t=0$ then we have for the mean :

$$(2)\quad Ey\ (t)=x_0 \exp\ (b+\tfrac{1}{2}c)\ t$$

The variance is :

$$(3)\quad \sigma^2_{(yt)}=[E\ y\ (t)]^2\ [exp\ (ct)-1].$$

We give in the following tables the data from which real per capita agricultural production and real per capita government expenditure on agriculture has been computed :

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TABLE 1

Net National Output from Agriculture at 1948-49 Prices

(in Rs. abja)*

Year	*t*	
1951-52	0	44·4
1952-53	1	46·0
1953-54	2	49·8
1954-55	3	50·3
1955-56	4	50·2
1956-57	5	52·5
1957-58	6	50·1
1958-59	7	55·6
1959-60	8	55·1
1960-61	9	59·0
1961-62	10	59·1
1962-63	11	58·0
1963-64	12	59·0

*abja=100 crores.

Source : Estimates of National Income 1948-49 to 1962-63, February 1964, issued by Central Statistical Organisation, Department of Satisticts, Cabinet Secretariat, Government of India.

TABLE 2

*Public Outlays for Agriculture in India**

Year	*t*	*Rs. crores*
1951-52	0	90·37
1952-53	1	102·82
1953-54	2	129·01
1954-55	3	177·52
1955-56	4	224·22
1956-57	5	151·34
1957-58	6	167·57
1958-59	7	189·99
1959-60	8	207·30
1960-61	9	237·00
1961-62	10	251·00
1962-63	11	307·00
1963-64	12	344·00

*The items of expenditure included are ; (*a*) Agriculture and Community Development : (*b*) Major and Medium Irrigation including Flood Control.

Source : (1) Government of India, *Third Five Year Plan.*

(2) Government of India, *The Third Plan Mid-term Appraisal*, p. 17.

Note. The figures upto 1955-56 include some expenditure on "Power" associated with multipurpose irrigation projects. See Third Five Year Plan, p. 738.

TABLE 3

Index Numbers of Wholesale Prices (All Commodities)

1952-53 = 100

Year	*t*	
1951-52	0	118·0
1952-53	1	100·0
1953-54	2	101·2
1954-55	3	89·6
1955-56	4	92·5
1956-57	5	105·3
1957-58	6	108·4
1958-59	7	112·9
1959-60	8	117·1
1960-61	9	124·9
1961-62	10	125·1
1962-63	11	127·9
1963-64	12	135·3

Source : Reserve Bank of India Bulletin.

TABLE 4

Population (in crores)

Year	*t*	*Popln.*
1951-52	0	36·36
1952-53	1	37·00
1953-54	2	37·68
1954-55	3	38·39
1955-56	4	39·13
1956-57	5	39·91
1957-58	6	40·74
1958-59	7	41·59
1959-60	8	42·48
1960-61	9	43·42
1961-62	10	44·38
1962-63	11	45·36
1963-64	12	46·40

TABLE 5

Real Per Capita Agricultural Production *Real Per Capita Government Expenditure on Agriculture*

t	Xt	$log\ Xt$	Gt
0	1·0347	0·0149	0·0211
1	1.2430	0·0945	0·0278
2	1·3063	0·1158	0·0338
3	1·4621	0·1620	0·0516
4	1·3870	0·1421	0·0619
5	1·2488	0·0966	0·0360
6	1·1347	0·0551	0·0379
7	1·1842	0 0734	0·0405
8	1·1076	0·0445	0·0417
9	1·0881	0·0366	0·0437
10	1·0647	0·0273	0·0452
11	1·0000	0·0000	0·0529
12	0·9401	1·9731	0·0548

Assume now that $x=x_0$ with probability one at the point $t=0$, also that the observations are evenly spaced from $t=0$ to $t=n$ ($n+1$ observations). The maximum likelihood estimates of the parameters of the process are then given by the formulae :

(4) $$\hat{b}=\left(\sum_{t=1}^{n} \log x_t-\log x_{t-1}\right)/n$$

(5) $$\hat{c}=\left[\sum_{t=1}^{n} (\log x_t-\log x_{t-1})^2\right]/n-\hat{b}^2$$

The asymptotic variances of the estimates are given by :

(6) $$\sigma^2_{\hat{b}}=c/nt$$

(7) $$\sigma^2_{\hat{c}}=2c^2/n$$

Since both variances go to zero as n and t tend to infinity the estimates are seen to be consistent.

Fitting a lognormal diffusion process to the data by the methods indicated we obtain the following fitted values and predictions, under the assumption indicated in Table 6.

TABLE 6

t	$\log x^*_t$	X^*_t
0	= 0·0149	1·0350
1	0·0121	1·0280
2	0·0093	1·0220
3	0·0065	1·0150
4	0·0037	1·0090
5	0·0009	1·0020
6	$\bar{1}$·9981	0·9956
7	$\bar{1}$·9953	0·9893
8	$\bar{1}$·9925	0·9828
9	$\bar{1}$·9897	0·9766
10	$\bar{1}$·9869	0·9703
11	$\bar{1}$·9841	0·9640
12	$\bar{1}$·9813	0·9579
13	$\bar{1}$·9785	0·9517
14	$\bar{1}$·9757	0·9456
15	$\bar{1}$·9729	0·9396
16	$\bar{1}$·9701	0·9335
17	$\bar{1}$·9673	0·9274
18	= $\bar{1}$·9645	0·9215

The predicted value of $\overset{*}{x}_{18}$, *i e.* real per capita agricultural income for 1969-70 is 0·9215. Using the estimated variance, we might compute 95% confidence or fiducial limits, which are : 0·708 and 1·113.

Consider now the influence of an external factor *e g.* REAL public expenditure PER CAPITA in the year t, G_t.

Define :

(8) $H_t = G_0 + G_1 + \ldots + G_t$

the total expenditure in the period zero to t.

The effect of this expenditure will only be felt after p years. Then the transition probability will be :

$$(9)\ f(s, x\ ;\ t, y) = exp\ \frac{-\{\log y - \log x - b_0\ (t-s) - b_1\ (H_{t-p} - H_{s-p})/2c(t-s)^2\}}{y\sqrt{2\pi c(t-s)}}$$

This is the conditional probability, that the variable in question (*e.g.* REAL agricultural production PER CAPITA) will have the value y at the time t if it had the value x at the time s.

Assuming again that x has the value of x_p with probability one at the time $t=p$, we derive the following formula for the mean value :

$$(10)\quad E\,[y\,(t)] = x_p\ exp\ [(b_0 + \tfrac{1}{2}c)(t-p) + b_1 H_{t-p}]$$

and for the variance :

$$(11)\quad \sigma^2_{y(t)} = [E\ y(t)]^2\ [exp\,(ct) - 1]$$

The maximum likelihood estimates of the parameters are now as follows :

$$(12)\quad \hat{b}_1 = \frac{\sum_{t=1}^{n-p} G_t \left(\log x_{t+p} - \log x_{t+p-1}\right) - \dfrac{\left[\sum_{t=1}^{n-p} \log x_{t+p} - \log x_{t+p-1}\right]\left[\sum_{t=1}^{n-p} G_t\right]}{n-p}}{\sum_{t=1}^{n-p} G_t^2 - \dfrac{\left[\sum_{t=1}^{n-p} G_t\right]^2}{n-p}}$$

$$(13)\quad \hat{b}_0 = \frac{\sum_{t=1}^{n-p} \log x_{t+p} - \log x_{t+p-1} - b_1 \sum_{t=1}^{n-p} G_t}{n-p}$$

$$(14)\quad \hat{c} = \sum_{t=1}^{n-p} (\log x_{t+p} - \log x_{t+p-1} - b_0 - b_1\, G_t)^2/(n-p)$$

Now the large sample variances of the estimates are given by :

$$(15)\quad \sigma^2_{\hat{b}_0} = c/(n-p)t$$

(16) $\sigma_{\hat{b}_1}^2 = ct/(n-p) \sum_{t=1}^{n-p} G_t$

(17) $\sigma_{\hat{c}}^2 = 2c^2/n$

It is again seen that these variances tend to zero as t and n tend to infinity and G_t is constant or increasing.

Hence under these circumstances the estimates are consistent.

TABLE 7

t	*log x'*	x'
4	0 1727	1·4880
5	0·1465	1·4020
6	0·1233	1·3280
7	0·1091	1·2850
8	0·1002	1·2600
9	0·0782	1·1980
10	0·0571	1·1400
11	0·0373	1·0900
12	0·0181	1·0420
13	$\bar{1}$·9999	0·9997
14	$\bar{1}$·9825	0·9605
15	$\bar{1}$·9690	0·9311
16	$\bar{1}$·9565	0·9046

The fitted values of a log-normal diffusion process under the assumptions indicated are given in Table 7. We have also made a forecast for the year 1969-70 under the following alternative assumptions :

(α) same real per capita government expenditure in the intervening years as in 1963-64 (β) an increase by one half. (γ) a doubling (δ) two and a half times ($\in$) three times the expenditure of 1963-64. The forecasts and 95% limits are given in the following table 8.

TABLE 8

	Est x'_{18}	*95% lts*		
α)	0·8541	0·5952	—	1·1130
β)	0·9107	0·6347	—	1·1867
γ)	0·9712	0·6768	—	1·2656
δ)	1·0350	0,7212	—	1·3488
ε)	1·1040	0·7694	—	1·4386

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